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Cosmological Einstein–Maxwell Instantons and Euclidean Supersymmetry: Anti–Self–Dual Solutions.

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Abstract

We classify super–symmetric solutions of the minimal $N = 2$ gauged Euclidean supergravity in four dimensions. The solutions with anti–self–dual Maxwell field give rise to anti–self–dual Einstein metrics given in terms of solutions to the $SU(\infty)$ Toda equation and more general three–dimensional Einstein–Weyl structures. Euclidean Kastor–Traschen metrics are also characterised by the existence of a certain super covariantly constant spinor.

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1 Introduction

The bosonic sector of $N = 2$ supergravity (SUGRA) in four dimensions coincides with the Einstein–Maxwell theory. In [28] all solutions which admit a supercovariantly constant spinor have been found.

In this work we shall classify supersymmetric solutions of Euclidean Einstein–Maxwell equations with non–zero cosmological constant. It will be shown that the field equations in various branches of our classification reduce to the Einstein–Weyl system in three dimensions [18, 19, 7] which is integrable by twistor construction. Some of the Euclidean solutions arise from analytic continuations of real Lorentzian solutions - for example the Euclidean analogs of Kastor–Traschen metrics [20] belong to this class - while others do not have Lorentzian counterparts. In particular all solutions with anti–self–dual Maxwell field belong to the latter class. It turns out (Proposition 2.1) that the anti–self–duality of the Maxwell field implies the conformal anti–self–duality (ASD) of the Weyl tensor. In this paper we shall focus on constructing all solutions belonging to this ASD class. The non anti–self–dual solutions will be constructed in [8]. Some of these have a Lorentzian counterpart [5, 3, 24, 16, 14, 15].

In the second part of this introduction we shall discuss the Euclidean Einstein–Maxwell theory and explain the origin of various sign choices in the Euclidean signature. In Section 2 we shall use the two–component spinor calculus to classify all supersymmetric solutions. The Killing spinor equations (2.9) contain a continuous parameter and we shall show that the Killing spinor gives rise to a Killing vector only for one special value of this parameter. In this symmetric case the metric is given in terms of solutions to the $SU(\infty)$ Toda equation (Proposition 2.2). For all other values of the parameter the solutions do not in general admit an isometry. They do however admit a conformal retraction (Proposition 2.3 and Proposition 2.4). In Section 3 we shall characterise the Euclidean Kastor–Traschen solutions by the existence of a supercovariantly constant spinor with certain additional properties (Proposition 3.1). The solutions constructed in this section are not anti–self–dual.

There are several motivations for studying Euclidean gauged SUGRA solutions. From the differential geometric perspective the supersymmetric solutions constructed in Proposition 2.3 and Proposition 2.4 provide examples of anti–self–dual Einstein metrics. The point is that the energy–momentum tensor of the ASD Maxwell field vanishes and the Maxwell equations decouple from the Einstein equations. In Euclidean Quantum gravity instantons provide semi–classical description of black hole creations and in the cosmological context this has been implemented in [17, 27, 22, 23]. Finally solutions of ungauged ($\Lambda = 0$) SUGRA can be used to construct supersymmetric solutions of Lorentzian minimal SUGRA theories in five and higher dimensions [10, 1, 2]. It remains to be seen whether solutions to the gauged $D = 4$ Euclidean SUGRA admit such lifts.

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1.1 Euclidean Einstein–Maxwell equations

Consider Lorentzian Einstein–Maxwell equations possibly with non-vanishing cosmological constant

$$G_{ab} + 6\Lambda g_{ab} = -T_{ab}, \qquad dF = 0, \qquad d * F = 0, \tag{1.1}$$

where

$$T_{ab} = \frac{1}{2} g_{ab}|F|^2 - 2F_{ac}F_b{}^c$$

is the Maxwell energy–momentum tensor[1] and $|F|^2 = F_{cd}F^{cd}$. Swapping the electric and magnetic fields, i. e. replacing F by its Hodge dual $*F$ maps solutions to solutions as the Lorentzian Maxwell E-M tensor is unchanged by this transformation. This can be easily seen in the two–component spinor notation [25] where

$$T_{ab} = 2\phi_{AB}\overline{\phi}_{A'B'}$$

and the duality transformation is $\overline{\phi}_{A'B'} \to i\overline{\phi}_{A'B'}$ and $\phi_{AB} \to -i\phi_{AB}$. This is no longer the case in the Riemannian signature where

$$T_{ab} = 2\phi_{AB}\tilde{\phi}_{A'B'}$$

and the spinors ϕ_{AB} and $\tilde{\phi}_{A'B'}$ are independent. The transformation $F \to *F$ entails to $\tilde{\phi}_{A'B'} \to \tilde{\phi}_{A'B'}$ and $\phi_{AB} \to -\phi_{AB}$, thus $T_{ab} \to -T_{ab}$. This duality transformation can be used to 'fix wrong signs' arising from the analytic continuation of a Lorentzian solution.

As an example consider the Reissner–Nordström–de Sitter space–time

$$g = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad A = -Q\frac{dt}{r}, \tag{1.2}$$

where

$$V(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - 2\Lambda r^2,$$

$m \geq 0$ and Q are constants and $F = dA$. Now continue this analytically to the Riemannian signature setting $t = i\tau$ and assuming that r lies between the middle roots of the quartic

[1] Our conventions follow Penrose and Rindler [25]: $[\nabla_a, \nabla_b]V^d = R_{abc}{}^dV^c, R_{ab} = R_{acb}{}^c = 6\Lambda g_{ab} - 2\Phi_{ab}$, where Φ_{ab} is the traceless Ricci tensor. Using these conventions in the Riemannian settings implies that the hyperbolic space H^4 has $\Lambda > 0$ and the four-sphere S^4 has $\Lambda < 0$.

$r^2V(r) = 0$. We encounter an immediate problem as the potential A is now purely imaginary. There does not seem to be a universally accepted resolution of this problem, and the way in which one proceeds is dictated by an overall aim of the analytic continuation. According to Hawking and Ross [17] one should, at least in the Quantum–Mechanical context, accept that the electrically charged solution has imaginary gauge potential. Alternatively, especially if our interest lies in classical solutions, one could replace A by iA which is real. However this continuation changes the overall sign of T_{ab} and leads to a 'wrong' coupling between the gravitational and electromagnetic fields. The coupling can now be 'made right' by replacing $F \to *F$, resulting in the Maxwell field $F = - * d(Qr^{-1}d\tau)$.

In this discussion we used 'wrong' and 'right' in inverted commas, as the energy of the Riemannian Maxwell field is not positive definite, and (unlike in the Lorentzian case) the positivity can not be used to fix the relative sign between G_{ab} and T_{ab}. The cosmological constant in our example doesn't change under the analytic continuation. In Section 3 we shall however see a different class of examples (Kastor–Traschen cosmological multi black holes [20]) where the analytic continuation leads to a real solution only if Λ changes sign: asymptotically de Sitter Lorentizan solutions become asymptotically hyperbolic Riemannian solutions.

To avoid making the various sign choices we shall simply look for real solutions of the Euler–Lagrange equations arising from the Lagrangian density

$$\mathcal{L} = \sqrt{g}(R + \gamma|F|^2 + \delta),$$

where γ and δ are real constants. The cosmological constant can then be read–off from δ and the sign of the Maxwell energy–momentum tensor can be adjusted if necessary replacing F by its Hodge dual as explained above.

2 Supersymmetric solutions with ASD Maxwell field

Let the two–form F be an anti–self–dual (ASD) Maxwell field on a Riemannian four–manifold (M, g), i. e.

$$dF = 0, \qquad *F = -F.$$

We shall make use of an isomorphism

$$\mathbb{C} \otimes TM \cong \mathbb{S} \otimes \mathbb{S}'$$

where the complex rank–two vector bundles $\mathbb{S}, \mathbb{S}'$ (called spin-bundles) over M are equipped with parallel symplectic structures $\varepsilon, \varepsilon'$ such that $g = \varepsilon \otimes \varepsilon'$. We use the standard convention [25, 7] in which spinor indices are capital letters, unprimed for sections of $\mathbb{S}$ and primed for sections of $\mathbb{S}'$. The two component spinor formalism will be adapted to the Riemannian signature, where the spinor conjugation preserves the type of spinors. Thus if $\alpha_A = (p, q)$ we can define $\hat{\alpha}_A = (\overline{q}, -\overline{p})$ so that $\hat{\hat{\alpha}}_A = -\alpha_A$. This Hermitian conjugation induces a positive inner product

$$\hat{\alpha}_A\alpha^A = \varepsilon_{AB}\hat{\alpha}^A\alpha^B = |p|^2 + |q|^2 \, .$$

We define the inner product on the primed spinors in the same way. Here ε_{AB} and $\varepsilon_{A'B'}$ are covariantly constant symplectic forms with $\varepsilon_{01} = \varepsilon_{0'1'} = 1$. These are used to raise and lower spinor indices according to $\alpha_B = \varepsilon_{AB}\alpha^A, \alpha^B = \varepsilon^{BA}\alpha_A$, and similarly for primed spinors.

The spinor decomposition of the Riemann tensor is

$$\begin{aligned} R_{abcd} &= \psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \psi_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} \\ &\quad +\Phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} + \Phi_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} \\ &\quad +2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'B'}\varepsilon_{C'D'} - \varepsilon_{AB}\varepsilon_{CD}\varepsilon_{A'D'}\varepsilon_{B'C'}), \end{aligned} \tag{2.3}$$

where ψ_{ABCD} and $\psi_{A'B'C'D'}$ are ASD and SD Weyl spinors which are symmetric in their indices, $\Phi_{A'B'CD} = \Phi_{(A'B')(CD)}$ is the traceless Ricci spinor and $\Lambda = R/24$ is the cosmological constant.

Making use of the isomorphism $\Lambda^2_- \cong \mathbb{S}\odot\mathbb{S}$ we can write $F_{ab} = \phi_{AB}\varepsilon_{A'B'}$ where the symmetric spinor $\phi_{AB} = \hat{\phi}_{AB}$ satisfies the ASD Maxwell equations

$$\nabla^A{}_{A'}\phi_{AB} = 0.$$

Consider the Killing spinor equations [13, 28]

$$\begin{aligned} \nabla_{AA'}\alpha_B + c_0 A_a\alpha_B + (c_1\phi_{AB} + c_2\varepsilon_{AB})\beta_{A'} &= 0, \\ \nabla_{AA'}\beta_{B'} + c_3 A_a\beta_{B'} + c_4\varepsilon_{A'B'}\alpha_A &= 0, \end{aligned} \tag{2.4}$$

where A_a is a real one–form and $c_0, \ldots, c_4$ are some constant coefficients which we shall now determine. Differentiating (2.4) covariantly, commuting covariant derivatives and using the spinor Ricci identities

$$\nabla^A{}_{(A'}\nabla_{B')A}\alpha_B + \Phi_{A'B'AB}\alpha^A = 0, \tag{2.5}$$

$$\nabla^{A'}{}_{(A}\nabla_{B)A'}\beta_{B'} + \Phi_{A'B'AB}\beta^{A'} = 0, \tag{2.6}$$

$$\nabla^{A'}{}_{(A}\nabla_{B)A'}\alpha_C + \psi_{ABCD}\alpha^D - 2\Lambda\alpha_{(A}\varepsilon_{B)C} = 0, \tag{2.7}$$

$$\nabla^A{}_{(A'}\nabla_{B')A}\beta_{C'} + \psi_{A'B'C'D'}\beta^{D'} - 2\Lambda\beta_{(A'}\varepsilon_{B')C'} = 0, \tag{2.8}$$

leads to the compatibility conditions: Equations (2.5) give

$$\Phi_{ABA'B'} = 0, \qquad c_0\nabla^A{}_{(A'}A_{B')A} = 0, \qquad c_0 = c_3.$$

Equations (2.6) give $F = 2dA$ if $\nabla^A{}_{(A'}A_{B')A} = -\tilde{\phi}_{A'B'}, \nabla^{A'}{}_{(A}A_{B)A'} = -\phi_{AB}$ or

$$c_3 = -c_1c_4.$$

Equations (2.7) give

$$c_2c_4 = -\Lambda.$$

Finally equations (2.8) give

$$\psi_{A'B'C'D'} = 0$$

so we deduce

Proposition 2.1 *A Riemannian four–manifold which admits a solution to the Killing spinor equations* (2.4) *with anti–self–dual Maxwell field F is anti–self–dual and Einstein.*

The case $c_0 = c_1 = c_3 = 0$ leads to some non–trivial solutions in $(2,2)$ signature, but not in $(4,0)$ so we shall not consider it. If $c_0 \neq 0$, then we can redefine A_a, ϕ_{AB} and $\beta_{A'}$ by rescalings to get rid of some constants. Set $c_0 = -ce^{i\theta}$, where c and θ are real. The resulting equations are

$$\begin{aligned} \nabla_{AA'}\alpha_B &= e^{i\theta}A_a\alpha_B + \Big(\frac{e^{i\theta}}{\Lambda}\phi_{AB} - \varepsilon_{AB}\Big)\beta_{A'} \\ \nabla_{AA'}\beta_{B'} &= e^{i\theta}A_a\beta_{B'} + \Lambda\varepsilon_{A'B'}\alpha_A, \end{aligned} \tag{2.9}$$

together with equations for spinor conjugates

$$\begin{aligned} \nabla_{AA'}\hat{\alpha}_B &= e^{-i\theta}A_a\hat{\alpha}_B + \Big(\frac{e^{-i\theta}}{\Lambda}\phi_{AB} - \varepsilon_{AB}\Big)\hat{\beta}_{A'} \\ \nabla_{AA'}\hat{\beta}_{B'} &= e^{-i\theta}A_a\hat{\beta}_{B'} + \Lambda\varepsilon_{A'B'}\hat{\alpha}_A. \end{aligned} \tag{2.10}$$

2.1 $\theta = \pi/2$ and $SU(\infty)$ Toda equation

Now we shall consider the case $\theta = \pi/2$ and show that the resulting metric is the most general ASD Einstein metric with symmetry, and can be found from solutions to $SU(\infty)$ Toda equation.

Proposition 2.2 *Let the Riemannian four–manifold (M, g) admit a solution to the Killing spinor equations* (2.9) *with $\theta = \pi/2$ such that $F_{ab} = \phi_{AB}\varepsilon_{A'B'}$ is an ASD Maxwell field with $F = 2dA$. Then g satisfies ASD Einstein equations with non–zero Λ. Moreover g admits a Killing vector and local coordinates (x, y, z, τ) can be chosen so that*

$$g = \frac{1}{z^2}\Big(V(dz^2 + e^u(dx^2 + dy^2)) + V^{-1}(d\tau + \omega)^2\Big), \tag{2.11}$$

where $u = u(x, y, z)$ is a solution of the $SU(\infty)$ Toda equation

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0, \tag{2.12}$$

the function V is given by $-4\Lambda V = zu_z - 2$, and ω is a one–form such that

$$d\omega = -V_x dy \wedge dz - V_y dz \wedge dx - (Ve^u)_z dx \wedge dy. \tag{2.13}$$

We have already shown that g is ASD and Einstein. Once we establish the existence of a symmetry, we could refer to results of Tod [29] and Przanowski [26] to deduce the canonical form of the metric (2.11). In the proof below we shall however give a direct derivation of this form using the Killing spinor equations.

Proof. Define two real non–zero functions $U, \widetilde{U}$ by

$$U = (\varepsilon_{AB}\hat{\alpha}^A\alpha^B)^{-1}, \qquad \widetilde{U} = (\varepsilon_{A'B'}\hat{\beta}^{A'}\beta^{B'})^{-1}. \tag{2.14}$$

Consider a complex tetrad of one–forms

$$K_a = i(\hat{\alpha}_A\beta_{A'} + \alpha_A\hat{\beta}_{A'}), \quad X_a = \hat{\alpha}_A\beta_{A'} - \alpha_A\hat{\beta}_{A'}, \quad Z_a = \alpha_A\beta_{A'}. \tag{2.15}$$

The one–forms $X = X_a e^a, K = K_a e^a$ are real and the one–form $Z = Z_a e^a$ is complex. Using the Killing spinor equations (2.9) and their conjugations (2.10) we find

$$\nabla_a K_b = \varepsilon_{A'B'}(\Lambda(\hat{\alpha}_A\alpha_B + \alpha_A\hat{\alpha}_B) - \frac{i}{\Lambda\widetilde{U}}\phi_{AB}) - \varepsilon_{AB}(\hat{\beta}_{A'}\beta_{B'} + \beta_{A'}\hat{\beta}_{B'}) \tag{2.16}$$

so that $\nabla_{(a}K_{b)} = 0$ and K is a Killing vector. Moreover we find

$$dX = 0, \qquad Z \wedge dZ = 0$$

and deduce existence of a local coordinate system $(\tau, \zeta, q, \overline{q})$ on M such that

$$K^a\nabla_a = \sqrt{2}\frac{\partial}{\partial\tau}, \quad X = \sqrt{2}d\zeta, \quad Z = \frac{1}{\sqrt{2}}pdq$$

for some complex–valued function $p = p(\zeta, q, \overline{q})$. Therefore the one–form dual to the Killing vector is $K = \Omega(d\tau + \omega)$, where Ω and ω are a function and a one–form on the space of orbits of K in M. Using

$$\varepsilon_{AB} = U(\hat{\alpha}_A\alpha_B - \hat{\alpha}_B\alpha_A), \quad \varepsilon_{A'B'} = \widetilde{U}(\hat{\beta}_{A'}\beta_{B'} - \hat{\beta}_{B'}\beta_{A'}) \tag{2.17}$$

we find the metric to be

$$g = \varepsilon_{AB}\varepsilon_{A'B'}e^{AA'}e^{BB'} = U\widetilde{U}(d\zeta^2 + |p|^2dqd\overline{q} + \Omega^2(d\tau + \omega)^2).$$

Finally using $K_aK^a = 2(U\widetilde{U})^{-1}$ and setting $q = x + iy$, $|p|^2 = e^\phi$ where $\phi = \phi(x, y, \zeta)$ is a real–valued function yields

$$g = U\widetilde{U}(e^\phi(dx^2 + dy^2) + d\zeta^2) + \frac{1}{U\widetilde{U}}(d\tau + \omega)^2.$$

Now we need to find equations for $\phi, U, \widetilde{U}$ and ω. Using the Killing spinor equations (2.9) gives

$$\nabla_a\Big(\frac{1}{\widetilde{U}}\Big) = \Lambda X_a, \quad \nabla_a\Big(\frac{1}{U}\Big) = -\frac{1}{\Lambda}\phi_A{}^C K_{CA'} + X_a. \tag{2.18}$$

Therefore $\widetilde{U} = (\sqrt{2}\Lambda\zeta)^{-1}$, where we absorbed the integration constant into the definition of ζ. Defining a coordinate $z = \zeta^{-1}$ and setting

$$U = \sqrt{2}\Lambda z V, \quad \phi = u + 4\log\zeta$$

where $V = V(x, y, z), u = u(x, y, z)$ yields the final form of the metric (2.11). We now use (2.18) to find

$$\phi_{AB} = \frac{2\Lambda}{|K|^2} K_B^{A'} \nabla_{AA'} \Big(\frac{1}{U} - \frac{\Lambda}{\widetilde{U}} \Big),$$

and substitute this to (2.16). This yields $-4\Lambda V = z u_z - 2$, where u satisfies the SU(∞) Toda equation (2.12), and (2.13) holds.

$\square$

Note that the rescaled metric $\hat{g} = z^2 g$ is of the form given by the LeBrun ansatz [21] because V satisfies the linearised $SU(\infty)$ Toda equation. Therefore $\hat{g}$ is Kähler with vanishing Ricci scalar. Scalar–flat Kähler metrics are also solutions to Einstein–Maxwell equations in the Riemannian signature [11], where the self–dual (SD) and anti–self–dual (ASD) parts of the Maxwell field are given by the Kähler form Ω and (half of) the Ricci form ρ respectively

$$F = \Omega + \frac{\rho}{2}.$$

Note that $\Omega \in \Lambda^2_+$ and $\rho \in \Lambda^2_-$. Thus there exist two conformally related metrics: one non-supersymmetric $\hat{g}$ which solves Euclidean ungauged supergravity equations (Einstein–Maxwell with $\Lambda = 0$), and one supersymmetric g which solves the gauged supergravity (Einstein–Maxwell with $\Lambda \neq 0$).

2.2 $\theta = 0$ and the hyperCR equation

The ASD Einstein metrics corresponding to $\theta \neq \pi/2$ in (2.9) do not in general admit a continuous symmetry. In this subsection we shall find a general local form of the metric in the case when $\theta = 0$.

Proposition 2.3 *Let the Riemannian four–manifold (M, g) admit a solution to the Killing spinor equations (2.9) with $\theta = 0$ such that $F_{ab} = \phi_{AB}\varepsilon_{A'B'}$ is an ASD Maxwell field with $F = 2dA$. Then g satisfies ASD Einstein equations with $\Lambda > 0$. Moreover a local coordinate ψ can be chosen so that*

$$g = \frac{\Lambda}{8} \sinh{(2\psi)}^2 h + \frac{2}{\Lambda}(d\psi - \coth{(\psi)}\omega)^2, \qquad F = 2\, d(\coth{(\psi)}^2 \omega) \tag{2.19}$$

where $h = \boldsymbol{e}_1^2 + \boldsymbol{e}_2^2 + \boldsymbol{e}_3^2$ is a 3-metric, the ψ–independent one–forms $(\boldsymbol{e}_i, \omega)$ satisfy $\partial_\psi \lrcorner\, \boldsymbol{e}_i = \partial_\psi \lrcorner\, \omega = 0$,

$$\begin{aligned} d\boldsymbol{e}_1 &= -2\omega \wedge \boldsymbol{e}_1 - \Lambda \boldsymbol{e}_2 \wedge \boldsymbol{e}_3, \\ d\boldsymbol{e}_2 &= -2\omega \wedge \boldsymbol{e}_2 - \Lambda \boldsymbol{e}_3 \wedge \boldsymbol{e}_1, \\ d\boldsymbol{e}_3 &= -2\omega \wedge \boldsymbol{e}_3 - \Lambda \boldsymbol{e}_1 \wedge \boldsymbol{e}_2, \end{aligned} \tag{2.20}$$

and

$$d\omega = \Lambda *_h \omega, \tag{2.21}$$

where $*_h$ is the Hodge operator of h.

Proof. The ASD Einstein equations follow from the integrability conditions as we have already explained.

The gauge freedom

$$\alpha_A \to e^f \alpha_A, \quad \beta_{A'} \to e^f \beta_{A'}, \quad A \to A - df, \qquad \text{where} \quad f : M \longrightarrow \mathbb{R}$$

can be used to set $\widetilde{U} = 1$. Consider a tetrad (2.15), so that with our gauge choice

$$g_{ab} = \frac{U}{2}(4Z_{(a}\overline{Z}_{b)} + K_a K_b + X_a X_b)$$

and $X_a X^a = K_a K^a = 2Z_a \overline{Z}^a = 2U^{-1}$ and all other inner product vanish.

The condition $d(\widetilde{U}^{-1}) = 0$ implies

$$A_a = \frac{\Lambda}{2} X_a.$$

We also find

$$X^a \nabla_a (U^{-1}) = 2U^{-1}(\Lambda U^{-1} - 1),$$

so that if τ is a local coordinate for which $X^a \nabla_a = \partial/\partial\tau$ then

$$U = \Lambda(1 + e^{2\tau} c^2), \tag{2.22}$$

where c is a local function on M independent on τ. Now we use the Killing spinor equations (2.9) and (2.17) to find

$$\begin{aligned} dZ &= (-UX + i(U - \Lambda)K) \wedge Z \\ dK &= -UX \wedge K + 2i(U - \Lambda) Z \wedge \overline{Z}, \end{aligned} \tag{2.23}$$

so regarding $X = \partial/\partial\tau$ as a vector field

$$\mathcal{L}_X Z = -2Z, \qquad \mathcal{L}_X K = -2K$$

where $\mathcal{L}_X$ denotes the Lie derivative along the vector field $X = X^a \nabla_a$. Therefore we can set

$$Z = e^{-2\tau} \widetilde{Z}, \quad K = e^{-2\tau} \widetilde{K}$$

where $\widetilde{Z}$ and $\widetilde{K}$ are one–forms which Lie–derive along X. The one form X_a is given by $X = 2U^{-1}(d\tau + \Omega)$, where Ω is a one–form which in general can depend on τ. We now have to consider two cases

1. $U = \Lambda$, which corresponds to vanishing of the function c in (2.22). Now

$$d\widetilde{Z} = -2\Omega \wedge \widetilde{Z}, \quad d\widetilde{K} = -2\Omega \wedge \widetilde{K},$$

so that Ω is independent on τ. Taking the exterior derivatives of these equations gives the integrability condition $d\Omega = 0$. Therefore locally there exist τ–independent real valued functions (ϕ, x, y, z) such that

$$\Omega = d\phi, \quad \widetilde{Z} = \frac{1}{2}e^{-2\phi}(dx + idy), \quad K = e^{-2\phi}dz.$$

Finally setting $\tilde{\tau} = \tau + \phi$ gives the hyperbolic metric

$$g = \frac{\Lambda}{2}e^{-4\tilde{\tau}}(dx^2 + dy^2 + dz^2) + \frac{2}{\Lambda}d\tilde{\tau}^2 \tag{2.24}$$

and the vanishing Maxwell field $F = 0$. This metric has $\Lambda > 0$ which is consistent with our curvature conventions.

2. Now assume $U \neq \Lambda$. Equations (2.23) imply

$$\begin{aligned} d\widetilde{Z} &= (-2\Omega + i\Lambda c^2\widetilde{K}) \wedge \widetilde{Z} \\ d\widetilde{K} &= -2\Omega \wedge \widetilde{K} + 2i\Lambda c^2\widetilde{Z} \wedge \overline{\widetilde{Z}}. \end{aligned} \tag{2.25}$$

We can redefine coordinates to set $c = 1$. To see it put

$$\widetilde{Z} = \frac{c^{-2}}{2}(\mathbf{e}_1 + i\mathbf{e}_2), \quad \widetilde{K} = c^{-2}\mathbf{e}_3, \quad \tilde{\tau} = \tau - \log c, \quad \omega = \Omega + d\log c. \tag{2.26}$$

Then the metric is given by

$$g = \Lambda(1 + e^{2\tilde{\tau}})\Big(\frac{1}{2}e^{-4\tilde{\tau}}h + \frac{2}{\Lambda^2(1 + e^{2\tilde{\tau}})^2}(d\tilde{\tau} + \omega)^2\Big),$$

where $h = \mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2$. Substituting (2.26) into (2.25) gives the system (2.20) for the one–forms $\mathbf{e}_i$ .The Maxwell field is given by

$$F = d\Big(\frac{2\omega}{1 + e^{2\tilde{\tau}}}\Big)$$

and the anti–self–duality condition $F = -*F$ yields (2.21). This is also the integrability condition for (2.20). Setting $\psi = -\text{arctanh}(\sqrt{1 + e^{2\tilde{\tau}}})$ yields the form of the metric and the Maxwell field given in the statement of the Proposition.

□

Remarks

- Making an analytic continuation $\psi \to i\psi$ and changing the sign of Λ leads to an ASD Einstein metric given in terms of trigonometric (rather than hyperbolic) functions. Setting $\Lambda = -4$ yields
$$g = \frac{1}{4}\sin^2(2\psi)h + \frac{1}{4}(d\psi + \cot\psi\,\omega)^2$$

- A three–manifold admitting a system of one–forms $(\mathbf{e}_i, \omega)$ satisfying equations (2.20) and (2.21) admits a hyper–CR Einstein–Weyl (EW) structure [12]. There is a well–known construction [19] which associates ASD conformal structures with symmetry to any EW structure. Proposition 2.3 reveals another connection between the hyperCR EW structures and ASD four–manifolds, where the Einstein metric in an ASD conformal class does not admit a symmetry.

- In [9] it was shown how to reduce the hyperCR conditions (2.20) and (2.21) to a single second–order integrable PDE (which therefore plays a role analogous to the $SU(\infty)$ Toda equation) for one function of three variables.

The metric (2.19) degenerates at $\psi = 0$ but this degeneracy can be absorbed into a conformal factor as
$$g = \sinh(2\psi)^2\hat{g}, \quad \text{where} \quad \hat{g} = \frac{\Lambda}{8}h + \frac{2}{\Lambda}(d\chi + e^{-2\chi}\sinh(2\chi)\,\omega)^2$$
and $\chi = -\text{arctanh}(e^{2\psi})$. The conformal structure will therefore be regular if the pair (h, ω) is. An example is provided by the Berger sphere, where[2]
$$h = (\sigma_1)^2 + (\sigma_2)^2 + \Lambda^2(\sigma_3)^2, \qquad \omega = \frac{1}{2}\Lambda\sqrt{1-\Lambda^2}\sigma_3,$$
where $0 < \Lambda \leq 1$ and σ_i are the left–invariant one–forms on S^3 satisfying
$$d\sigma_1 = \sigma_2 \wedge \sigma_3, \quad d\sigma_2 = \sigma_3 \wedge \sigma_1, \quad d\sigma_3 = \sigma_1 \wedge \sigma_2.$$

2.3 General θ and interpolating Einstein–Weyl equations

Finally we shall analyse the Killing spinor equations (2.9) and (2.10), where the parameter θ is allowed to take arbitrary values. Similarly to the cases $\theta = \pi/2$ and $\theta = 0$ we shall find that the space–time admits a local fibration over a three–dimensional manifold with an Einstein–Weyl structure. The relevant Einstein–Weyl structure has arisen in [9] as the most general symmetry reductions of ASD Ricci–flat equations by a conformal Killing vector. It contains both the $SU(\infty)$ and hyperCR equations as special cases. The class of ASD Einstein metrics characterised in the following Proposition does not in general admit an isometry (or

[2]The equations (2.20) hold, but not in the 'obvious' frame $\mathbf{e}_1 = \sigma_1, \mathbf{e}_2 = \sigma_2, \mathbf{e}_3 = \Lambda\sigma_3$. See [6] for relevant formulae.

a conformal isometry) unless $\theta = \pi/2$. Instead it will be shown to admit an ASD conformal retraction in a sense of [18] and [4] (in [4] it is referred to a conformal submersion. The metrics from Proposition 2.3 belong to the class described in Theorem IX in this reference. The metrics characterised by the proposition below appear to be new).

Proposition 2.4 *Let (M, g) be a Riemannian four–manifold which admits a solution of the Killing spinor equations* (2.9) *and* (2.10) *such that the two–form $F = 2dA$ is anti–self–dual. Then g is anti–self–dual and Einstein with $\Lambda \neq 0$ and locally is of the form*

$$g = \frac{2}{\Lambda}\Big(\frac{e^{2\cos\theta\tau}}{1+e^{2\cos\theta\tau}}\Big)\Big(d\tau - \omega + \frac{1}{2}\Lambda\tan\theta\, e^{2\cos\theta\tau}\ \boldsymbol{e}_3\Big)^2 + \frac{\Lambda}{2}e^{4\cos\theta\tau}(1+e^{-2\cos\theta\tau})h \qquad (2.27)$$

where $h = \boldsymbol{e}_1^2 + \boldsymbol{e}_2^2 + \boldsymbol{e}_3^2$ is a 3-metric, the $\tau-$ independent one–forms $(\boldsymbol{e}_i, \omega)$ satisfy $\partial_\tau \lrcorner\, \boldsymbol{e}_i = \partial_\tau \lrcorner\, \omega = 0$, and

$$\begin{aligned} d\boldsymbol{e}_3 &= -2\cos\theta\omega\wedge\boldsymbol{e}_3 - \Lambda\cos\theta\,\boldsymbol{e}_1\wedge\boldsymbol{e}_2 \\ d(\boldsymbol{e}_1 + i\boldsymbol{e}_2) &= (-2e^{-i\theta}\omega - ie^{-i\theta}\Lambda\boldsymbol{e}_3)\wedge(\boldsymbol{e}_1 + i\boldsymbol{e}_2). \end{aligned} \qquad (2.28)$$

Proof. To establish this result we shall use the same strategy as in the proof of Proposition 2.3. The calculations are further complicated by the presence of θ but the main steps are as before: use the gauge freedom to set $\widetilde{U}$ to a constant, explore the Killing spinor equations to solve for the Maxwell potential A and use the Frobenius theorem to construct a triad of one–forms out of the Killing spinors defining a conformal structure on a three–manifold.

Using (2.9) and (2.10) we find

$$\begin{aligned} \nabla_a U^{-1} &= 2U^{-1}\cos\theta A_a + \alpha_A\hat{\beta}_{A'} - \hat{\alpha}_A\beta_{A'} + \frac{1}{\Lambda}\phi_{AB}(e^{-i\theta}\alpha^B\hat{\beta}_{A'} - e^{i\theta}\hat{\alpha}^B\beta_{A'}), \\ \nabla_a\widetilde{U}^{-1} &= 2\widetilde{U}^{-1}\cos\theta A_a + \Lambda(\alpha_A\hat{\beta}_{A'} - \hat{\alpha}_A\beta_{A'}), \end{aligned}$$

where $U, \widetilde{U}$ are defined by (2.14). Use the gauge freedom in scalings of the spinors to set $\widetilde{U}$ to a constant. This gives an expression for A. Set

$$T^a = \frac{1}{\sqrt{2}}(e^{-i\theta}\alpha^A\hat{\beta}^{A'} - e^{i\theta}\hat{\alpha}^A\beta^{A'}),$$

and define a real one–form W by $\alpha_A\hat{\beta}_{A'} = \sqrt{2}^{-1}e^{i\theta}(T_a + iW_a)$, so that

$$g(W, W) = g(T, T) = (U\widetilde{U})^{-1}.$$

We also define two real one–forms $\mathbf{e}_1, \mathbf{e}_2$ by $Z = f\sqrt{2}^{-1}(\mathbf{e}_1 + i\mathbf{e}_2)$, where $Z^a = \alpha^A\beta^{A'}$ and f is some function. Now introduce a local coordinate τ such that $T^a\nabla_a = \partial/\partial\tau$ and so

$$T = (U\widetilde{U})^{-1}(d\tau + \alpha) \qquad (2.29)$$

for some one-form α which in general depends on τ . Calculating $T^a\nabla_a U^{-1}$ yields

$$\frac{\partial}{\partial\tau}U^{-1} = \sqrt{2}\cos\theta\, U^{-1}(\widetilde{U}^{-1} - \Lambda U^{-1}).$$

There are two cases to consider: If $U = \lambda\widetilde{U} = \text{const}$, then $\phi_{AB} = 0$, so $dA = 0$ and without loss of generality we can set $A = 0$ in some gauge. Using an argument analogous to the one leading to (2.24) we find that g is a hyperbolic metric with constant scalar curvature (this has $\Lambda > 0$ in our conventions). Otherwise we have

$$\tau = \frac{1}{\mu}\left(\ln\frac{\widetilde{U}}{(U - \Lambda\widetilde{U})}\right) + \ln(c), \quad \text{where} \quad \mu = \sqrt{2}\widetilde{U}^{-1}\cos\theta, \quad c = \text{const}.$$

The Killing spinor equations give

$$dZ = 2e^{i\theta}A \wedge Z - 2UZ \wedge \bar{Y} - 2\Lambda\widetilde{U}Z \wedge Y,$$

where $Y_a = \alpha_A\hat{\beta}_{A'}$. Set $Z = f(\tau)Z_0$, where $\dot{f}/f = \sqrt{2}\exp(-i\theta)\widetilde{U}^{-1}$ so that

$$dZ_0 = (\sqrt{2}e^{-i\theta}\widetilde{U}^{-1}\alpha + i\Big(\frac{\sqrt{2}\Lambda\widetilde{U}}{\cos\theta} - \sqrt{2}e^{-i\theta}U\Big)W) \wedge Z_0.$$

We also find

$$dT + idW = \sqrt{2}e^{-i\theta}(iU\,T \wedge W + (U - \Lambda\widetilde{U})|f|^2 Z_0 \wedge \bar{Z}_0)$$

and

$$dW = \sqrt{2}\cos\theta(U\,T \wedge W - (U - \Lambda\widetilde{U})|f|^2\mathbf{e}_1 \wedge \mathbf{e}_2).$$

Defining a one–form $\mathbf{e}_3$ by $W = g(\tau)\mathbf{e}_3$, where $g = g(0)\exp(\mu\tau)$ and substituting this in the expression for dW yields

$$d\mathbf{e}_3 = \sqrt{2}\widetilde{U}^{-1}\cos\theta\;\alpha \wedge \mathbf{e}_3 - \sqrt{2}\beta\cos\theta\mathbf{e}_1 \wedge \mathbf{e}_2, \tag{2.30}$$

where $\beta = \widetilde{U}|f(0)|^2/(cg(0))$ is a constant. Similarly the expression for dZ_0 yields

$$d(\mathbf{e}_1 + i\mathbf{e}_2) = (\sqrt{2}e^{-i\theta}\widetilde{U}^{-1}\alpha + i\Big(\frac{2\Lambda\widetilde{U}}{\cos\theta} - 2e^{-i\theta}U\Big)g\mathbf{e}_3) \wedge (\mathbf{e}_1 + i\mathbf{e}_2). \tag{2.31}$$

We now have to establish the dependence of α on τ. The Killing spinor equations yield

$$\nabla_{(a}T_{b)} = 2\cos\theta A_{(a}T_{b)} + \sqrt{2}^{-1}(\widetilde{U}^{-1} + \Lambda U^{-1})\cos\theta g_{ab}.$$

Therefore $\mathcal{L}_T h = \theta\, h$, where h_{ab} is the part of g_{ab} orthogonal to T^a and the last equality is valid modulo T. Thus T_a is a conformal retraction. Moreover this retraction is ASD in the sense of [4] as dA is ASD. We further find

$$\mathcal{L}_T T_a = \sqrt{2}\cos\theta(\widetilde{U}^{-1} - \Lambda U^{-1})T_a - \sqrt{2}U^{-1}\sin\theta\, W_a.$$

Finally using (2.29) gives

$$\alpha = -\omega + \Lambda \widetilde{U}^2 g(0) \tan\theta \; e^{\mu\tau} \; \mathbf{e}_3,$$

where ω is some τ–independent one–form orthogonal to $\partial/\partial\tau$. To obtain equations (2.28) in the Proposition we substitute this expression into (2.30) and (2.31), and make the following choices for the so far unspecified constants

$$\widetilde{U} = \frac{1}{\sqrt{2}}, \quad g(0) = c\Lambda, \quad \beta = \sqrt{2}^{-1}\Lambda$$

which is consistent if we also chose $(f(0))^2 = (g(0))^2$. Note that c can also be chosen arbitrarily by adding a constant to τ. To obtain the formulae in the Proposition we set $c = \Lambda^{-1}$. The metric g is given by

$$g = U\widetilde{U}(T^2 + Z^2 + |f|^2((\mathbf{e}_1)^2 + (\mathbf{e}_2)^2)),$$

where

$$U = \frac{\widetilde{U} + c\widetilde{U}\Lambda e^{\mu\tau}}{ce^{\mu\tau}}.$$

This, with our choice of constants, gives (2.27).

□

Remark. A three–dimensional Einstein–Weyl structure consists of a conformal structure $[h]$ and a torsion–free connection D such that

$$D_i h_{jk} = \nu_i h_{jk}, \quad R_{ij} + \frac{1}{2}\nabla_{(i}\nu_{j)} + \frac{1}{4}\nu_i\nu_j = \mathcal{W} h_{jk}, \qquad i, j, k = 1, 2, 3,$$

where ν is a one–form, ∇_i and R_{ij} are respectively the Levi–Civita connection and the Ricci tensor of $h \in [h]$ and $\mathcal{W}$ is a function which can be read–off by taking a trace of both sides of the second equation. In [9] it was shown that

$$h = (\mathbf{e}_1)^2 + (\mathbf{e}_2)^2 + (\mathbf{e}_3)^2, \quad \nu = -4\omega\cos\theta - 4\Lambda\sin\theta\mathbf{e}_3, \quad \Lambda = \text{const}$$

satisfies the Einstein–Weyl equations if the triad $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ satisfies (2.28). Moreover the Einstein–Weyl structure arising this way is the most general symmetry reduction of hyper–Käher metric in four dimension by a conformal symmetry.

3 Euclidean Kastor–Traschen solutions

In this section we shall drop the ASD condition on the Maxwell field so that

$$F_{ab} = \phi_{AB}\varepsilon_{A'B'} + \tilde{\phi}_{A'B'}\varepsilon_{AB}.$$

Thus the Killing spinor equations (2.4) are replaced by

$$\begin{aligned}\nabla_{AA'}\alpha_B + c_0 A_a \alpha_B + (c_1\phi_{AB} + c_2\varepsilon_{AB})\beta_{A'} &= 0,\\ \nabla_{AA'}\beta_{B'} + c_3 A_a \beta_{B'} + (c_5\tilde{\phi}_{A'B'} + c_4\varepsilon_{A'B'})\alpha_A &= 0,\end{aligned}$$

where an additional term involving $\tilde{\phi}_{A'B'}$ is present. We now impose the integrability conditions (2.5)–(2.8) and proceed as before, but use the Einstein–Maxwell condition

$$\Phi_{ABA'B'} = 2\phi_{AB}\tilde{\phi}_{A'B'}.$$

We find that

$$c_0 = c_3 = \frac{1}{L}, \quad c_2 = \frac{1}{2L}c_1, \quad c_5 = -\frac{2}{c_1}, \quad c_4 = -\frac{1}{Lc_1},$$

where

$$\Lambda = \frac{1}{2L^2}$$

is a cosmological constant (which at this stage can be positive or negative if L is real or imaginary respectively). A constant rescaling of $\beta_{A'}$ can be used to set c_1 to any given non–zero constant. To achieve a symmetric form of the equations we replace $\beta_{A'}$ by $\sqrt{2}\beta_{A'}/c_1$ which results in $c_1 = \sqrt{2}$. The final form of the Killing spinor condition is

$$\begin{aligned}\nabla_{AA'}\alpha_B + \frac{1}{L}A_a\alpha_B + \sqrt{2}(\phi_{AB} + \frac{1}{2L}\varepsilon_{AB})\beta_{A'} &= 0,\\ \nabla_{AA'}\beta_{B'} + \frac{1}{L}A_a\beta_{B'} - \sqrt{2}(\tilde{\phi}_{A'B'} + \frac{1}{2L}\varepsilon_{A'B'})\alpha_A &= 0.\end{aligned} \tag{3.32}$$

We conclude that the non-ASD case is more 'rigid' than the ASD one. The Killing spinor equations (2.9) with ASD Maxwell field contain one essential parameter θ. If the Maxwell field is not ASD (or SD) the integrability conditions fix all the parameters in terms of the cosmological constant. Further analysis depends on the sign of the cosmological constant. If $\Lambda < 0$, the resulting metric admits a Killing vector $K_a = i(\hat{\alpha}_A\beta_{A'} + \alpha_A\hat{\beta}_{A'})$ and is given by a Riemannian analogue of the Caldarelli–Klemm solution [5, 3]. If $\Lambda > 0$ the symmetry is not present in general, and the metric is a Riemannian version of the solutions obtained in [16, 24].

In the proposition below we shall characterise Riemannian Kastor–Traschen solutions [20] as those where the ratio of norms of the spinors α_A and $\beta_{A'}$ is a constant. We define two real non–zero functions $U, \widetilde{U}$ by

$$U = (\varepsilon_{AB}\hat{\alpha}^A\alpha^B)^{-1}, \qquad \widetilde{U} = (\varepsilon_{A'B'}\hat{\beta}^{A'}\beta^{B'})^{-1}.$$

as before. The gauge transformations $\alpha \to e^f\alpha, \beta \to e^f\beta$ where $f : M \longrightarrow \mathbb{R}$ result in

$$U \to e^{-2f}U, \quad \widetilde{U} \to e^{-2f}\widetilde{U}$$

so that the ratio $U/\widetilde{U}$ is gauge invariant. In the rest of this section we shall assume $L = l \in \mathbb{R}$ and the cosmological constant is positive.

Proposition 3.1 *Let the Riemannian four–manifold* (M, g) *admit a solution to the Killing spinor equations* (3.32) *with* $\Lambda > 0$ *such that the gauge invariant condition*

$$\frac{U}{\widetilde{U}} = const$$

holds. Then (M, g) *is Einstein–Maxwell with*[3] $F = 2dA$ *and local coordinates* (x, y, z, T) *can be chosen so that*

$$g = \Big(\mathcal{U} + \frac{1}{l}T\Big)^2(dx^2 + dy^2 + dz^2) + \Big(\mathcal{U} + \frac{1}{l}T\Big)^{-2} dT^2, \quad F = 2\, d\Big(\frac{dT}{\mathcal{U} + l^{-1}T}\Big) \tag{3.33}$$

where $\mathcal{U} = \mathcal{U}(x, y, z)$ *satisfies the Laplace equation on* $\mathbb{R}^3$

$$\frac{\partial^2 \mathcal{U}}{\partial x^2} + \frac{\partial^2 \mathcal{U}}{\partial y^2} + \frac{\partial^2 \mathcal{U}}{\partial z^2} = 0.$$

Proof. The Einstein–Maxwell equations with $\Lambda > 0$ follow from the integrability conditions for (3.32). To find the local form of the metric first choose a gauge

$$U\widetilde{U} = 1.$$

Set $L = l \in \mathbb{R}$. The Killing spinor equations and their conjugates can be used to find

$$\begin{aligned}
\nabla_a(U^{-1}) &= -\frac{2}{l}A_a U^{-1} - \sqrt{2}\phi_A^B X_{BA'} - \frac{1}{l\sqrt{2}}X_a = 0,\\
\nabla_a(\widetilde{U}^{-1}) &= -\frac{2}{l}A_a \widetilde{U}^{-1} - \sqrt{2}\tilde{\phi}_{A'}^{B'} X_{AB'} - \frac{1}{l\sqrt{2}}X_a = 0.
\end{aligned}$$

These equations imply

$$U = \widetilde{U} = 1, \quad A_a X^a = -\frac{1}{\sqrt{2}}, \quad \tilde{\phi}_{A'}^{B'} X_{AB'} = \phi_A^B X_{BA'}.$$

The expression for A is found to be

$$A_a = -\frac{l}{2\sqrt{2}}(E_a + \frac{1}{l}X_a), \tag{3.34}$$

where $E_a = 2{\phi_A}^B X_{BA'} = X^b F_{ab}$. We also find

$$F = E \wedge X. \tag{3.35}$$

[3]The sign of the energy momentum tensor in this example is opposite to the one in (1.1). This sign can be changed if desired as explained in the Introduction by using the Maxwell field $F = 2 * d((\mathcal{U} + l^{-1}T)^{-1}dT)$.

Further application of the Killing spinor equations (3.32) gives

$$\begin{aligned} dZ &= (-l^{-1}\sqrt{2}X - 2l^{-1}A) \wedge Z, \\ dK &= (-l^{-1}\sqrt{2}X - 2l^{-1}A) \wedge K, \\ dX &= -2l^{-1}A \wedge X - \sqrt{2}F, \end{aligned} \tag{3.36}$$

and consequently

$$\mathcal{L}_X Z = -\frac{\sqrt{2}}{l}Z, \quad \mathcal{L}_X K = -\frac{\sqrt{2}}{l}K, \quad X \wedge dX = 0.$$

The integrability conditions for the first two equations in (3.36) come down to $d(X+\sqrt{2}A) = 0$, so that locally

$$X + \sqrt{2}A = d\gamma \tag{3.37}$$

for some function γ. Let τ be a local coordinate such that $X^a\nabla_a = \partial/\partial\tau$. We find that $X(\gamma) = 1$, so $\gamma = \tau + \tilde{\gamma}$, where $\tilde{\gamma}$ is a function which does not depend on τ. The one–form dual to $X = \frac{\partial}{\partial\tau}$ is $X = 2(d\tau + \Omega)$ for some one–form Ω. We can fix the falue of the constant $\tilde{\gamma}$ reabsorbing $d\tilde{\gamma}$ into the definition of Ω. Equations (3.36) now imply the existence of real local coordinates (x, y, z) such that

$$K = \frac{1}{\sqrt{2}}e^{-\sqrt{2}\tau/l}dz, \quad Z = \frac{1}{2\sqrt{2}}e^{-\sqrt{2}\tau/l}(dx + idy)$$

and combining (3.34) with $X + \sqrt{2}A = d\tau$ yields $\Omega = lE/2$ so that the metric is

$$g = e^{-2\sqrt{2}\tau/l}(dx^2 + dy^2 + dz^2) + 2\Big(d\tau + l\frac{E}{2}\Big)^2.$$

Using (3.37) and (3.35) we calculate $dX = ldE = -(\sqrt{2})^{-1}F$ and

$$\begin{aligned} \mathcal{L}_X E &= -\frac{1}{l\sqrt{2}}X \lrcorner F = -\frac{1}{l\sqrt{2}}X \lrcorner (E \wedge X) \\ &= \frac{\sqrt{2}}{l}E \end{aligned}$$

as $X \cdot X = 2$. Therefore $E = e^{\sqrt{2}\tau/l}\omega$, where ω is a one–form independent on τ. The condition $X \wedge dX = 0$ implies that $d\omega = 0$ so that locally $\omega = d\phi$, where $\phi = \phi(x, y, z)$ is some function. Using (3.37) we find

$$F = 2e^{\sqrt{2}\tau/l}d\phi \wedge d\tau.$$

Moreover

$$*F = \frac{4\sqrt{2}}{l} *_3 d\phi,$$

where $*_3$ is the Hodge operator of the flat 3-metric. Therefore the Maxwell equation $d * F = 0$ implies that ϕ is harmonic on $\mathbb{R}^3$ and

$$g = e^{-2\sqrt{2}\tau/l}(dx^2 + dy^2 + dz^2) + 2\Big(d\tau + \frac{l}{2}e^{\sqrt{2}\tau/l}d\phi\Big)^2.$$

To put the metric and the Maxwell field in the form (3.33) set

$$T = l\frac{\phi}{\sqrt{2}} - le^{-\sqrt{2}\tau/l}, \quad \mathcal{U} = -\frac{\phi}{\sqrt{2}}.$$

□

The solution (3.33) can be obtained as an analytic continuation of the Kastor–Traschen cosmological black holes [20]. This continuation requires the sign of cosmological constant to change.

Example. Setting $\mathcal{U} = 0$ in (3.33) gives the hyperbolic space. Consider $\mathcal{U} = m/R$, where m is a constant, and R is the radial coordinate on $\mathbb{R}^3$ so that the metric becomes

$$g = \Big(\frac{m}{R} + \frac{T}{l}\Big)^2\Big(dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)\Big) + \Big(\frac{m}{R} + \frac{T}{l}\Big)^{-2}dT^2. \tag{3.38}$$

This metric admits an isometry $(R, T) \to (c^{-1}R, cT)$ generated by the Killing vector

$$\mathcal{K} = T\frac{\partial}{\partial T} - R\frac{\partial}{\partial R}.$$

Introduce the coordinates (s, r) adapted to this isometry by

$$R = e^{-s/l}, \qquad T = l(r - m)e^{s/l}$$

so that $\mathcal{K}(s) = 0, \mathcal{K}(r) = 1$. Let $\psi = \psi(s, r)$ be a function such that

$$d\psi = ds + \frac{l(r - m)}{Vr^2}dr,$$

where

$$V = \frac{r^2}{l^2} + \Big(1 - \frac{m}{r}\Big)^2.$$

The metric then takes the form

$$g = Vd\psi^2 + V^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.39}$$

It closely resembles the analytic continuation of the Reissner–Nordström–de Sitter metric (1.2) described in the introduction in the extremal case $|Q| = m$. The difference between these two solutions lies in the sign of the cosmological constant. The extremal RNdS instanton with $\Lambda < 0$ has been named the lukewarm instanton in [27]. The conical singularities in the

metric are not present as the black–hole and the cosmological horizons have the same Hawking temperatures, i.e. $V(r_1) = V(r_2) = 0$ at these horizons and $|V'(r_1)| = |V'(r_2)|$. This instanton has been interpreted [27, 22] as describing a pair creation of non-extreme black holes in thermal equilibrium.

In our case $\Lambda > 0$. At $r \to \infty$ the metric approaches the constant curvature hyperbolic space. The limit $r \to 0$ is singular. This reflects the fact that the metric (3.38) is an analytic continuation of the Lorentzian Reissner–Nordström–de Sitter space–time, where the singularity is not hidden inside a horizon.

Example. If $\Lambda = 0$ and

$$\mathcal{U} = c + \sum_{m=1}^{N} \frac{a_m}{|\mathbf{x} - \mathbf{x}_m|}, \qquad a_1, \ldots, a_N, c = \text{const}$$

then (3.33) becomes the Majumdar–Papapetrou Einstein–Maxwell multi instanton [10]. The metric is asymptotically locally Robinson-Bertotti if $c = 0$, or asymptotically flat if $c \neq 0$ and T is periodic. In [10] it was shown how these instantons can be lifted to regular solitonic solutions to $\mathcal{N} = 2$ minimal five–dimensional supergravity. It remains to be seen whether the solutions (3.33) with non–zero Λ can also be uplifted to higher dimensions.

4 Conclusions

We have classified super–symmetric solutions of the minimal $N = 2$ gauged Euclidean supergravity in four dimensions, under the additional assumptions that the Maxwell field is anti–self–dual. The resulting metrics are Einstein, have anti–self–dual Weyl curvature and are given in terms of solutions to three–dimensional Einstein–Weyl equations. We have also found one class of examples corresponding to non ASD Maxwell field. These examples are Euclidean analogs of Kastor–Traschen cosmological metrics. The solutions constructed in the paper provide new examples of Einstein metrics in four dimensions. It remains to be seen whether they can be used to describe cosmological black hole creations and in the context of Euclidean Quantum Gravity.

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